Some variations on Planck's constant

preceded by

a few general considerations about physical constants

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The new SI and Planck's constant

Current definition: The kilogram is the unit of mass; it is equal to the mass of the international prototype of the kilogram.

Proposed definition: The kilogram is defined by taking the fixed numerical value of the Planck constant h to be $6.626070040 \times 10^{-34}$ when expressed in the unit $kg \cdot m^2 \cdot s^{-1}$.

Nothing to frighten theoretical physicists who have on their part decided for long that $h = 2\pi$…

I. A classification of physical constants

A conceptual inquiry into

— their meaning

— their role

— their fate

An epistemological perspective with various implications

— theoretical

— practical

— pedagogical

« On the Conceptual Nature of the Physical Constants », *Rivista Nuovo Cimento* 7, 187 (1977)

« The Importance of Being (a) Constant »

in *Problems in the Foundations of Physics* , p.237, Soc. Ital.di Fisica (1979)

Questions :

Are all the csts appearing in tables (m_p, q_e, h...) on the same footing ?

Why are there fundamental csts (h, c) in modern physics and not in classical physics ?

Are the conversion factors (k, J) fundamental csts ?

What is the meaning of putting a fundamental cst (h, c) equal to unity ?

The classification

Derived (secondary) constants

boiling point of water
lattice constant of silicium crystal
mass of the uranium atom

Fundamental (primary) constants

— type S : specific properties of “elementary” physical objects

mass of the Higgs boson

— type G : generic properties of classes of physical phenomena

coupling constants

— type U : universal constants scaling universal physical laws

Planck’s constant

The classification

Derived (secondary) constants

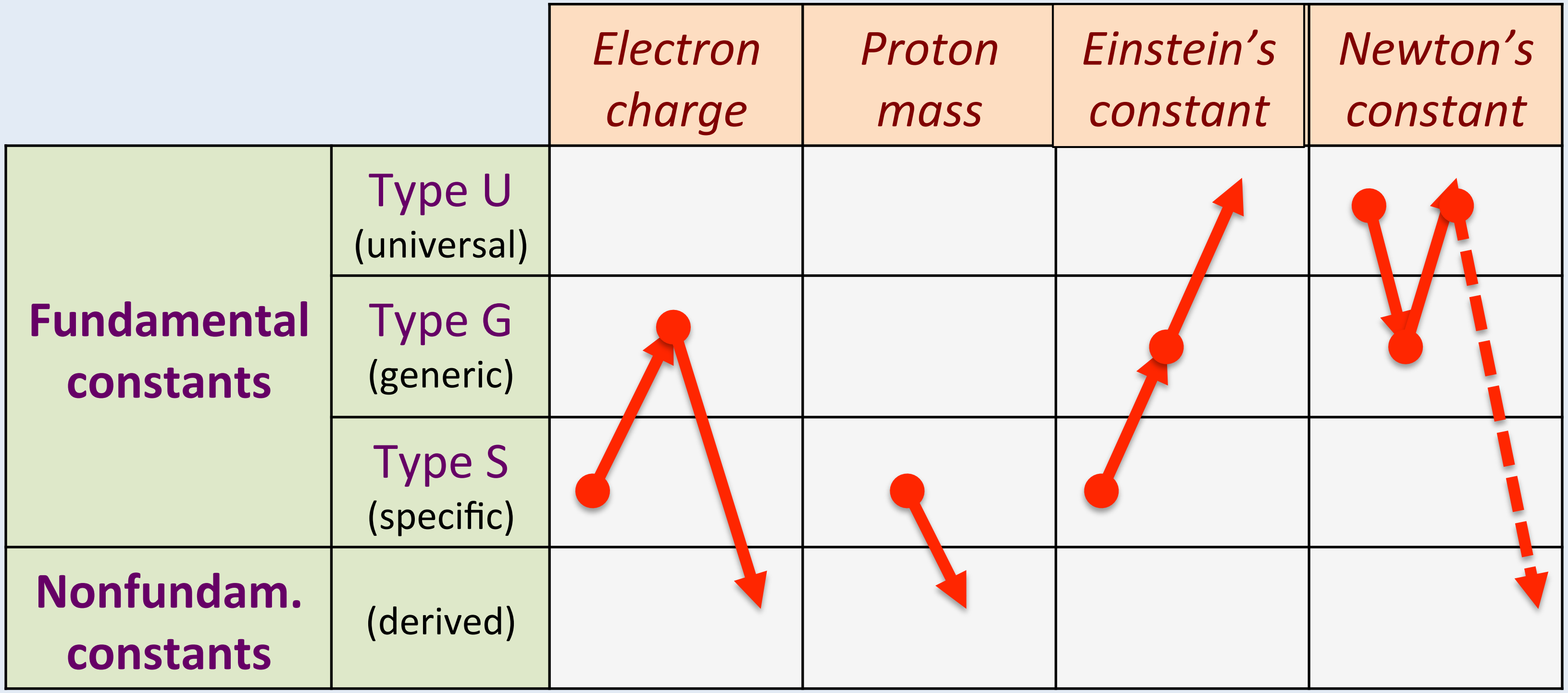
Fundamental (primary) constants

— type S : specific properties of "elementary" physical objects

— type G : generic properties of classes of physical phenomena

— type U : universal constants scaling universal physical laws

Not a closed and fixed classification !
Quite the contrary :
allows for a discussion of the variation of the constants,
as concerning their meaning, role, status, etc.



The epistemological fate of some physical constants

Universal constants as conceptual synthesizers

Thermodynamics

work W
heat Q

$W = JQ$ → energy

Einsteinian relativity

mass m
energy E

$E = mc^2$ → mass-energy

space Δx
time Δt

$\Delta s^2 = \Delta t^2 - c^{-2} \Delta x^2$ → space-time

The fading into unity of universal constants

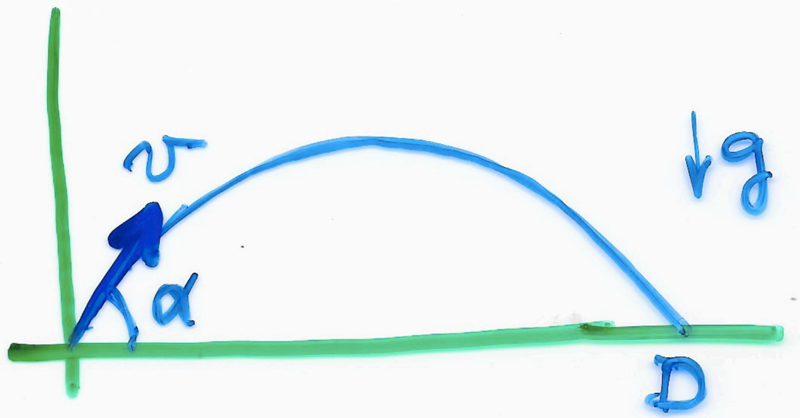
A historical fate :
from nobility (conceptual synthesis)
to domesticity (conversion factors)

Modern constants $E = mc^2$

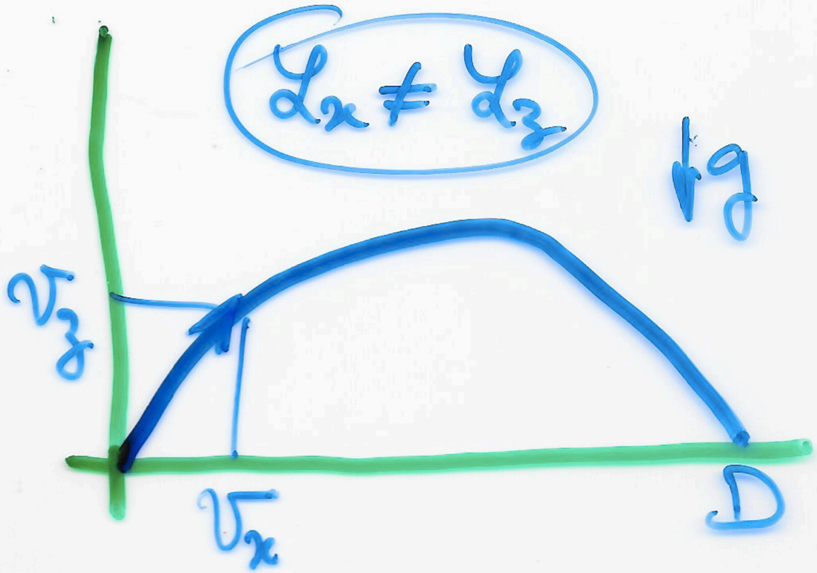
Classical constants $W = JQ$

Archaic constants $L_{vert} = \lambda L_{hor}$

The utility of archaic csts (Sommerfeld)



$$D = F(v, \alpha, g)$$
$$= f(\alpha)\,\frac{v^2}{g}$$

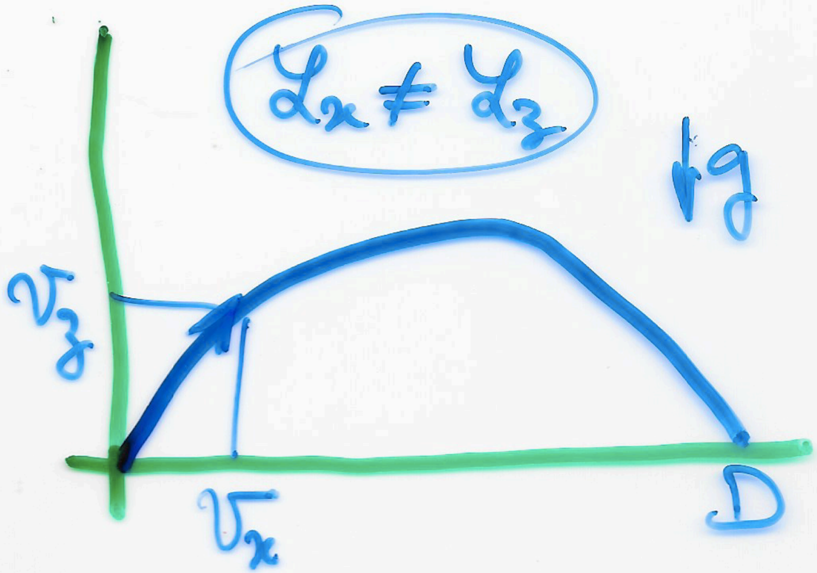


$$D = F(v_x, v_z, g)$$

$$D \rightarrow \mathcal{L}_x \qquad v_x \rightarrow \mathcal{L}_x \mathcal{T}^{-1} \qquad v_z \rightarrow \mathcal{L}_z \mathcal{T}^{-1} \qquad g \rightarrow \mathcal{L}_z \mathcal{T}^{-2}$$

$$D = C \, \frac{v_x v_z}{g}$$

$$= C \sin\alpha \cos\alpha \, \frac{v^2}{g}$$



$$D = F(v_x, v_z, g)$$

$D \rightarrow L_x$

$v_x \rightarrow L_x T^{-1}$

$v_z \rightarrow L_z T^{-1}$

$g \rightarrow L_z T^{-2}$

$$D = C \frac{v_x v_z}{g}$$

$$= C \sin\alpha \cos\alpha \frac{v^2}{g}$$

II. The Planck's constant

A. The birth of Planck's constant
B. Was Planck's constant really fundamental ?
C. The rise of Planck's constant to universality
D. h as the quantum of action
E. h as a standard of quanticity
F. Planck's h or Dirac's $\hbar = h/2\pi$?
G. Planck's constant at the macroscopic level

A. The birth of Planck's constant

Planck 1900 – 1901

« …the energy element should be proportional to the number of vibrations :

$$\varepsilon = h\nu$$

« …the energy distribution in the normal spectrum is

$$u = (8\pi h\nu^3/c^3)\ [\exp(h\nu/kT) - 1]^{-1}$$

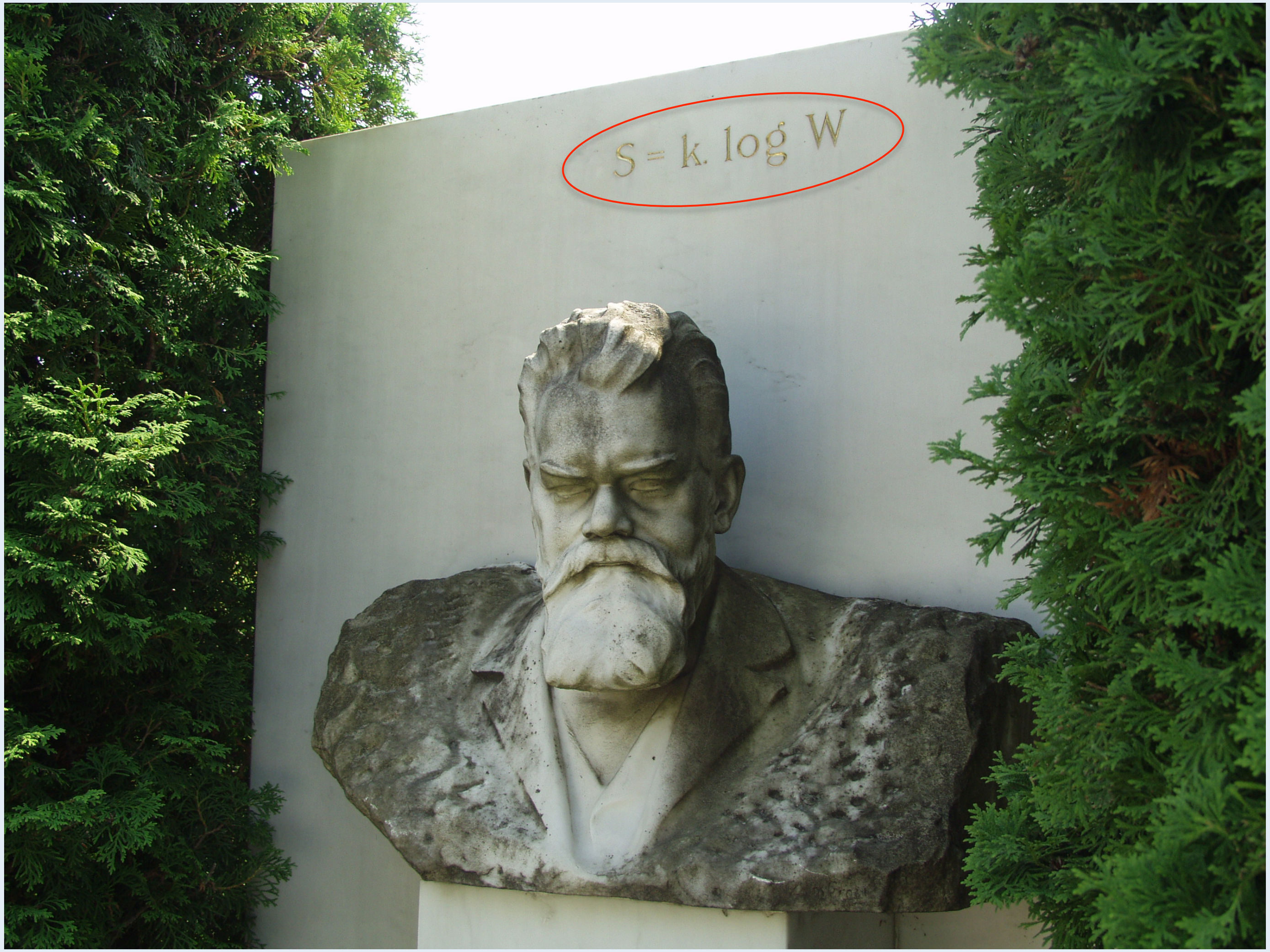
« …After the stationary energy distribution is thus determined using a constant *h*, we can find the corresponding temperature using a second constant of nature *k*… »

« …The values of both natural constants *h* and *k* may be calculated precisely with the help of measurements available… the values for the universal constants turn out :

$$h = 6.55 \cdot 10^{-27}\ \text{erg} \cdot \text{sec}$$

$$k = 1.346 \cdot 10^{-16}\ \text{erg/degree}$$

A short digression on Boltzmann's constant



But…

Boltzmann never wrote $S = k \log W$!

In his seminal 1877 paper*, his statement is that

« for each reversible change of state, the increase in the "permutability measure" [which is for us today the logarithm of the number of configurations] is equal to the entropy increment. »

No explicit formula.

In later papers, Boltzmann always used for "his" constant the expression R/N_0 (with R the perfect gas constant and N_0 the Avogadro number).

*"On the Relationship between the Second Fundamental Theorem of the Mechanical Theory of Heat and Probability Calculations Regarding the Conditions for Thermal Equilibrium",
Sitzungberichte der Kaiserlichen Akademie der Wissenschaften. Mathematisch-Naturwissen Classe. Abt. II, LXXVI 1877, pp 373-435

It is definitely Planck who decided to denote this constant by *k*, as he was the first to write

$$S = k \log W$$

in the same famous 1900-1901 papers where he introduced what would become "his" constant *h* — but he alaways complained that *k* should be called "the other Planck's constant".

This, by the way, explains the somewhat puzzling choice of the symbols *h* and *k*.

The fate of Planck's constant (step 0)

Fundamental constants	Type U (universal)	
	Type G (generic)	●
	Type S (specific)	
Nonfundam. constants	(derived)	

However, in his 1905 paper on the **photoelectric effect**, Einstein does not use Planck's constant !

Despite referring to Planck's 1901 article where the formula for the blackbody radiation is written with *h* and *k*, Einstein expresses it as

$$\rho_\nu = \alpha\nu^3\,[\exp(\beta\nu/T) - 1]^{-1}$$

And his crucial formula for the kinetic energy of the electron emitted is written

$$(R/N_0)\beta\nu - P$$

B. Was Planck's constant really fundamental ?

The fact is, that from 1905 to 1910, Einstein and others seem to have shared doubts on the really fundamental nature of Planck's constant.

The idea then was seriously discussed that *h* was in fact not a fundamental constant, characterizing the detailed mechanism of interaction between light and matter, reducing it to the status of a derived constant.

The fate of Planck's constant (step 1)

Fundamental constants	Type U (universal)	
	Type G (generic)	
	Type S (specific)	
Nonfundam. constants	(derived)	

By 1909, Einstein realized that the combination e^2/hc [using the modern notation $e^2 = q_e{}^2/4\pi\varepsilon_0$] was dimensionless, so that one could hope to "explain away" h by writing it as

$$h \sim e^2/c$$

within some numerical factor.

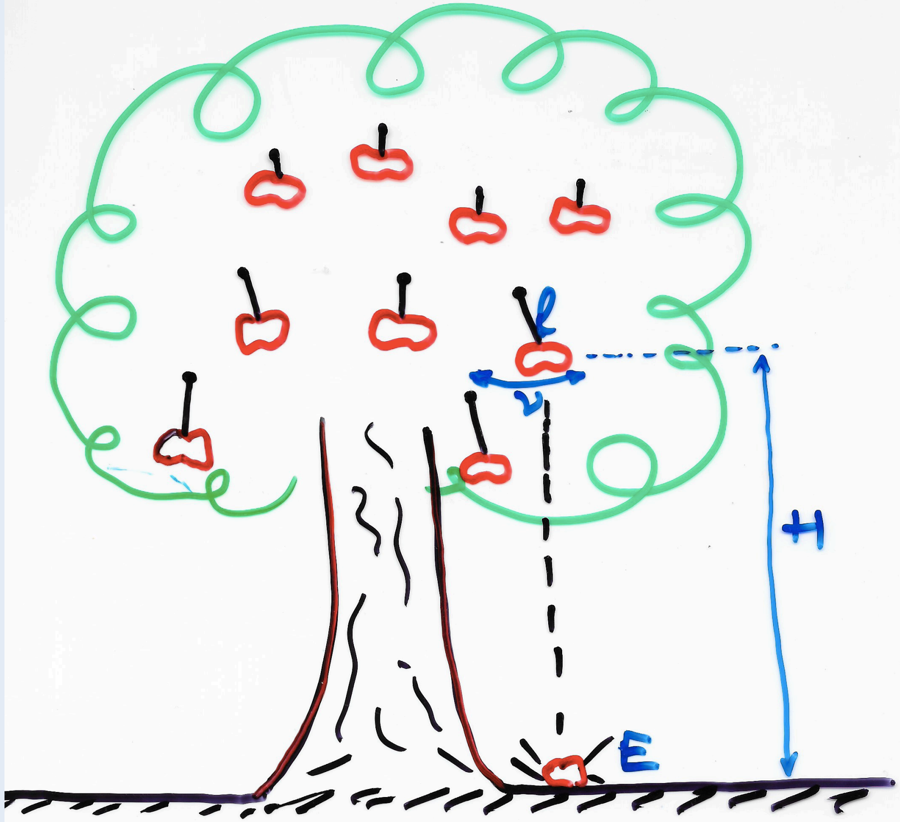
But, in a letter to Einstein (6 May 1909), Lorentz wisely objected that

> « I could imagine a factor of 4π or so intervening, but a factor 900, that is really too much. »

To which Einstein replied somewhat casually (23 May 1909) :

> « If only one could understand at least a little the relation $E = h\nu$! …The factor 900 missing in the non-rigorous relationship between h and e worries me as well, although introducing a factor such as $6(4\pi)^2$ would not be that extraordinary. For the dimensional argument seems very significant to me. »

The apple-tree model of Planck's formula (Born)



Suppose $\ell \propto H^{-2}$

then $\nu \propto H$

so that $E = mgH \propto \nu$

$$E = \text{"}h\text{"}\,\nu$$ (per apple)

Sound waves with frequency ν

($\rightarrow$ resonance) $\Rightarrow$ fall of N apples: energy $N\text{"}h\text{"}\nu$

C. The rise of Planck's constant to universality

Einstein 1907 (!) :

« Planck's theory of radiation and the theory of specific heat »

> « If the elementary structures that are to be assumed in the theory of energy exchange between radiation and matter cannot be perceived in terms of the current molecular kinetic theory, are we then not obliged alos to modify the theory for the other periodically oscillating structures considered in the molecular theory of solids ? In my opinion, the answer is not in doubt. »

Crucial extension of quantum hypotheses
to non-electromagnetic phenomena !

Cofirmed by the subsequent development of quantum theory.

The fate of Planck’s constant (step 2)

Fundamental constants	Type U (universal)	
	Type G (generic)	
	Type S (specific)	
Nonfundam. constants	(derived)	

D. *h* as the quantum of action

Planck noticed very early that *h* had the dimension of classical action. Then, at the 1911 Solvay conference, he introduced the idea that *h* was an « elementary quantum of action ».

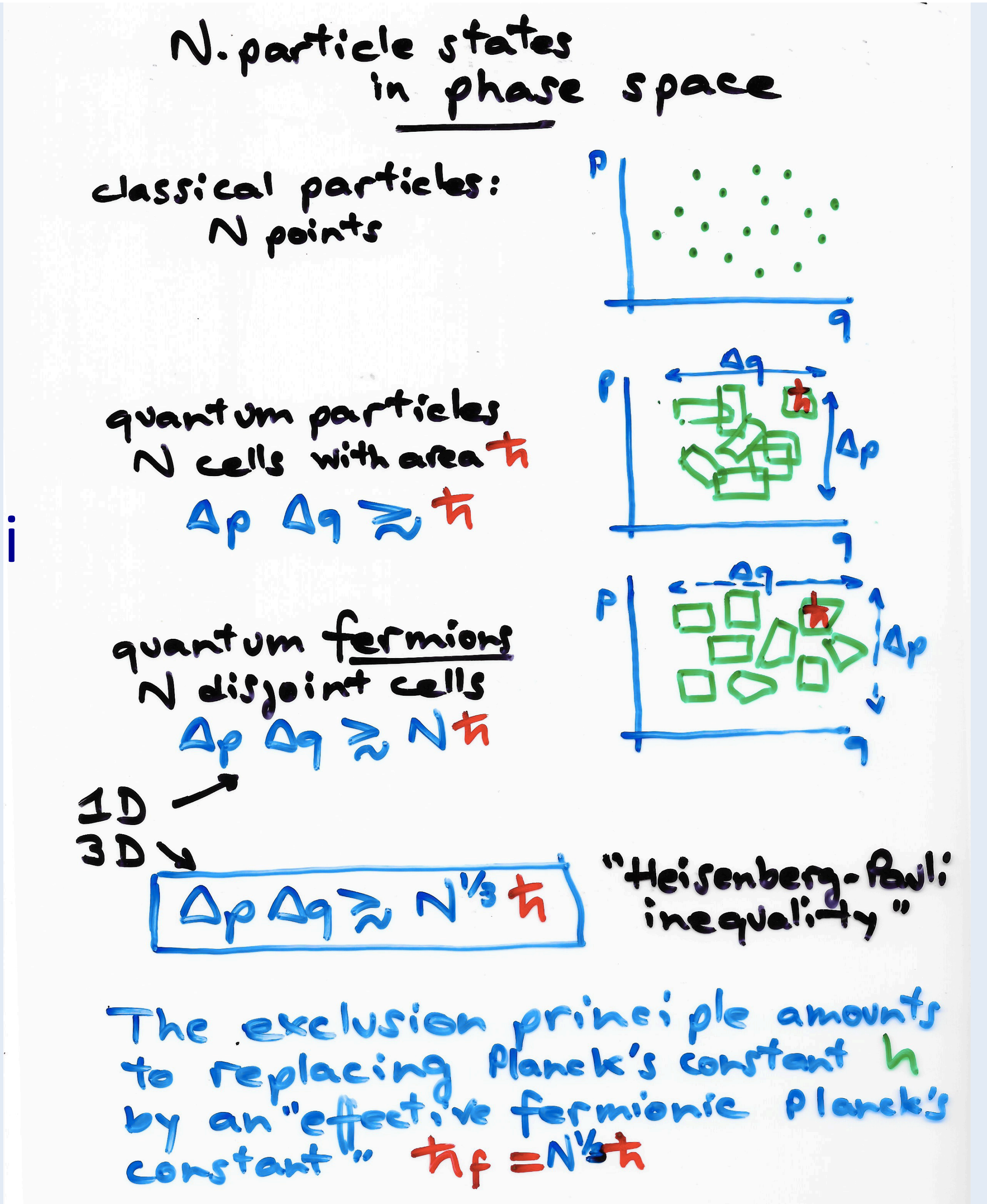
This was interpreted as establishing the existence of irreducible domains D in phase space, that is « cells », with finite extension *h* :

$$\iint_D dp\, dq = h$$

The idea was seized upon by Sommerfeld, who, inspired by what he called the « most fortunate » naming of h as the quantum of action, appealed to Hamilton's classical function of action to establish the rules of quantization in the so-called "Old Quantum Theory".

While too limited in scope (what is, after all, the "action" within quantum theory ?), this point of view still can be put too good use in heuristic considerations.

The Heisenberg-Pauli inequalities



Example: the ground state of Z-electrons atoms

$$H = \sum_k \frac{p_k^2}{2m} - \sum_k \frac{Ze^2}{r_k} + \sum_k \sum_l{}' \frac{e^2}{|r_k - r_l|}$$

$$E \simeq Z \frac{\tilde{p}^2}{2m} - Z^2 \frac{e^2}{\tilde{r}}$$

Heisenberg: $\tilde{p}\,\tilde{r} \gtrsim \hbar$

$$E \gtrsim Z \frac{\tilde{p}^2}{2m} - Z^2 \frac{e^2}{\hbar} \tilde{p}$$

$$\gtrsim \boxed{E_0 \simeq -Z^3 \frac{me^4}{\hbar^2}}$$ WRONG

Pauli: $\hbar \longrightarrow Z^{1/3} \hbar$

$$\boxed{E_0 \simeq -Z^{7/3} \frac{me^4}{\hbar^2}}$$ RIGHT

(Crude Thomas-Fermi approx.)

Saturation of Coulomb forces

For a system of N charges ⊕ and N charges ⊖, if the latter are fermions

$$\boxed{E_0 \simeq -N \frac{me^4}{\hbar^2}}$$

↓ thermodynamics, etc.

E. *h* as a standard of quanticity

With the advent of the new quantum mechanics in the 1920's, a novel meaning was attributed to Planck's constant.
Bohr and Heisenberg, in a joint paper on "matrix mechanics " at the 1927 Solvay conference delivered the following statement :

> « The real meaning of Planck's constant is this: it constitutes a universal gauge of the indeterminism inherent in the laws of nature owing to the wave-particle duality. »

One may propose a modern formulation, dispensing with the outdated terminology of "indeterminism" and "wave-particle duality" :

> *h* is a "standard of quanticity", which gives a general criterion for assessing the necessity of putting quantum theory to work.
> Namely, when considering a given physical phenomenon :
> — evaluate the relevant quantities with the dimension of an action
> — if they all are much larger than *h*, the classical theory yields a good approximation
> — if not, the recourse to quantum theory is compulsory.

F. *h* as the quantum concept synthesizer

Time — energy E, frequency ν > $E = h\nu$ — new concept (spectrum...)

Space — momentum p, wave number k > $p = hk$

Angle — Ang. mom. L, Ang. frequency μ > $L = h\mu$

$$\mu = \frac{1}{\alpha} \qquad \left(\nu = \frac{1}{T},\ k = \frac{1}{\lambda}\right)$$

with α = angular period = $\frac{2\pi}{m}$

F. Planck's h or Dirac's $\hbar=h/2\pi$?

Principle n° 0 of theoretical physics (Wheeler)

« The dimensionless constants of physics appearing in dimensional analysis are of order unity,

…if the characteristic magnitudes and parameters of the phenomenon are correctly chosen,

…with due allowance for exception. »

G. Planck's constant at the macroscopic level

The role of h is not restricted to the microscopic (atomic & subatomic domain).

For ordinary matter:

mass density $\frac{M}{(\hbar^2/me^2)^3}$ a few g cm^{-3}

electrochemical voltage

$\frac{1}{q_e} \frac{me^4}{\hbar^2}$ a few volts

h and the scale of life

Consider a planet with N atoms of average mass M.

Condition for life

1) Exchange of matter (Coulomb regime)

$$\frac{e^2}{a_0} \gtrsim G\frac{M \cdot NM}{N^{1/3} a_0} \qquad N \lesssim N_c$$

$$N_c = \left(\frac{e^2}{GM^2}\right)^{3/2}$$

$$R_c = \left(\frac{e^2}{GM^2}\right)^{1/2} a_0 \ *$$

2) Chemistry active

reactions $kT \simeq \frac{e^2}{a_0}$

atmosphere $kT \lesssim G\frac{M \cdot NM}{N^{1/3} a_0} \qquad N \gtrsim N_c$

3) Toughness of living beings (n atoms)
(no breaking on falling)

$$\mu g h < \text{breaking energy}$$

$$n M \frac{G N M}{(N^{1/3} a_0)^3} n^{1/3} a_0 < n^{2/3} \frac{e^2}{a_0}$$

Avogadro! $$n \lesssim N_c^{1/2} = \left(\frac{e^2}{GM^2}\right)^{3/4}$$

$(\lesssim 1\,m)$ $$\ell \lesssim \left(\frac{e^2}{GM^2}\right)^{1/4} \frac{\hbar^2}{me^2}$$

Long life to quantum t*h*eory!