Parity moments for a Poisson variable

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In counting random events which arrive at a mean rate ρ, it is always possible, by an appropriate choice of the measuring period t, to arrive at a mean number μ of registered events for which

$$\mu = \rho t < 1 . \tag{1}$$

In such a situation one may feel tempted - and this will sometimes be very advantageous indeed - to replace the traditional counting of the number of events by a determination of the frequency with which this number is even or odd.

The probability that the number of registered events k has odd parity, i.e. for $k = 2j + 1 = 1, 3, 5, \ldots$, is clearly

$$\Pi = \sum_{j=0}^{\infty} \text{prob}\,(2j + 1) . \tag{2}$$

Since the sum of all probabilities is normalized to unity, there is also the equivalent relation (for $k = 2j$)

$$\Pi = 1 - \sum_{j=0}^{\infty} \text{prob}\,(2j) , \tag{3}$$

and a combination of (2) and (3) leads to

$$\Pi = \frac{1}{2} \left[1 - \sum_{k=0}^{\infty} (-1)^k \,\text{prob}\,(k) \right] . \tag{4}$$

These three theoretical forms of the parity function Π are equivalent.

As shown previously, parities can be measured by an appropriate apparatus, normally in the form of an experimental ratio $\Pi = n_{odd}/n_{tot}$, where n_{odd} is the number of times the parity has been found to be odd among a total of n_{tot} trials.

An important special case is when the events under consideration can be assumed to follow Poisson statistics so that

$$\text{prob } (k) \; \equiv \; P_k = \frac{\mu^k}{k!} e^{-\mu} , \tag{5}$$

for k = 0, 1, 2, ... In this case the parity function, using (4), becomes

$$\Pi = \frac{1}{2} \left[1 - \sum_{k=0}^{\infty} \frac{(-\mu)^k}{k!} e^{-\mu} \right] = \frac{1}{2} (1 - e^{-2\mu}) . \tag{6}$$

This is a simple derivation of a basic result which has been obtained previously with some more effort [1]. For the situation (1) where μ is smaller than unity - the only case of interest for parity measurements - a series development may sometimes be used. One such approximation takes the form

$$\Pi = \mu \left[\sum_{j=0}^{\infty} \frac{(-2)^j}{(j+1)!} \mu^j \right]$$

$$= \mu (1 - \mu + \frac{2}{3} \mu^2 - \frac{1}{3} \mu^3 + \frac{2}{15} \mu^4 - \frac{2}{45} \mu^5 \pm ...) . \tag{7}$$

In using the results of parity measurements, normally for determining the rate of occurence for events which arrive as singles, i.e. without partner events which would form pairs, it often happens that we have to evaluate moments of experimental distributions. In particular, this need occurs in the determination of the corresponding dead-time corrections. The question then arises of how this can be achieved practically. It turns out that the so-called factorial moments provide a convenient tool.

For a positive integer number k, "falling factorials" of order r are defined by

$$(k)_r \; \equiv \; k \, (k-1) \, (k-2) \, ... \, (k-r+1) , \tag{8}$$

for r = 0, 1, 2, ..., using a shorthand notation borrowed from Riordan [2].

In consequence, a factorial moment of order r is given by

$$\overline{(k)_r} \; \equiv \; \sum_{k=r}^{\infty} (k)_r \, \text{prob } (k) . \tag{9}$$

For a Poisson variate k this leads to the surprisingly simple result

$$\overline{(k)_r} = \sum_k (k)_r P_k = \sum_k k(k-1) \dots (k-r+1) \frac{\mu^k}{k!} e^{-\mu}$$

$$= \sum_k \frac{\mu^k}{(k-r)!} e^{-\mu} = \mu^r e^{-\mu} \sum_{j=0}^{\infty} \frac{\mu^j}{j!} = \mu^r . \tag{10}$$

In the context of parities, what we have to evaluate are factorial "parity moments" of the form

$$(\Pi)_r \equiv \sum_{k=[r/2]}^{\infty} (2k+1)_r P_{2k+1} . \tag{11}$$

By taking advantage of the sum used in (6) and of the relation (10) we find

$$(\Pi)_r = \frac{1}{2} \left[\sum_k (k)_r P_k - \sum_k (k)_r (-1)^k P_k \right]$$

$$= \frac{1}{2} \left[\mu^r - \sum_k \frac{(-\mu)^k}{(k-r)!} e^{-\mu} \right] = \frac{1}{2} \left[\mu^r - (-\mu)^r e^{-\mu} \sum_j \frac{(-\mu)^j}{j!} \right]$$

$$= \frac{1}{2} \left[\mu^r - (-\mu)^r e^{-2\mu} \right] = \frac{\mu^r}{2} \left[1 - (-1)^r e^{-2\mu} \right] . \tag{12}$$

This is the required general expression for factorial parity moments of a Poisson variate.

If parity moments are to be evaluated, these are not necessarily of the type (11). Rather, one may have to deal with expressions of the form

$$\sum_k \left[\alpha_0 k^r + \alpha_1 k^{r-1} + \alpha_2 k^{r-2} + \dots \right] P_{2k+1} .$$

Hence, the expression in brackets has first to be expressed as a sum of falling factorials, i.e.

$$[\dots] = \sum_{j=0}^{r} \beta_j (2k+1)_j ,$$

so that each term can be transformed into a contribution to the corresponding parity function, i.e.

$$\Pi = \sum_j \beta_j \sum_k (2k+1)_j P_{2k+1} = \sum_j \beta_j (\Pi)_j . \tag{13}$$

These results thus solve the problem of evaluating parity moments for the general case, and in explicit form for a Poisson process.

References

[1] J.W. Müller: "Measurement of a dead time by correlation techniques", Rapport BIPM-89/10 (1989), Appendix

[2] J. Riordan: "An Introduction to Combinatorial Analysis" (Wiley, New York, 1958).

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